Selection of Intermediation

Michael Padilla

Stanford University

mtp@stanford.edu

Abstract

We consider the question of whether sellers, given a choice, would select intermediation over the standard mechanism in a blind portfolio auction (Padilla and Van Roy). Studying it within the framework of a 2-stage signaling game, we show that universal selection of intermediation across all seller types is a perfect Bayesian-Nash equilibrium and representative of all perfect Bayesian-Nash equilibria.

1 Introduction

In Padilla and Van Roy we demonstrated that in our model seller transaction costs improve when the blind portfolio auction is conducted in an intermediated fashion as opposed to the current standard format, with benefits on the order of 10% being realized under reasonable modeling assumptions. An important implicit assumption made was that in each case *all* sellers, regardless of their private type, chose either of the two mechanisms. In this report we consider a scenario where sellers come to the market and have the choice of conducting their auction via either the standard or intermediated mechanism, as would be the case in practice were both options available. When the standard mechanism is employed brokers do not receive any explicit information regarding the type q of the seller's portfolio, while when the intermediated mechanism is employed the intermediary is presented with this information. Note that due to the possibility of legal action and/or extreme reputational damage that would occur should the seller submit a false type to the intermediary, we make the assumption that sellers truthfully report their type to the intermediary when intermediation is selected.

We study the nature of any equilibria that may exist in this new game and show that universal selection of intermediation across all seller types is a perfect Bayesian-Nash equilibrium (Claim 4.1), and is indeed representative of all perfect Bayesian-Nash equilibria (Claim 4.2). Note that what follows is based on the same modeling framework developed in Padilla and Van Roy and in particular assumes the same broker preferences and valuations, repeated here for convenience:

Assumption 1.1. *Brokers exhibit constant absolute risk aversion. In other words, there exists a scalar $r > 0$ such that $u(v) = -\exp(-rv)$ for all v.*

Assumption 1.2. *For each n, the value of the portfolio to the nth broker is given by $v_n = v^* - q\theta_n$. The random variables $\theta_1, \ldots, \theta_N$ are iid with bounded support, denoted by Θ, and are independent of the random variable q, which we assume to also have bounded support. Both are positive with probability one and have non-zero measure. Further, the seller only observes q and each nth broker only observes θ_n.*

2 Bayesian Games and Perfect Bayesian-Nash Equilibria

To better understand the scenario in which each seller has a choice of which mechanism to employ, we consider it within the context of a *Bayesian multistage game with observable actions and independent types* in a manner analogous to the presentation in Fudenberg and Tirole (1991) and Osborne and Rubinstein (1994), which we will now briefly overview. In such a game, using the familiar transformation of a game of incomplete information to one of imperfect information by Harsanyi (1967), nature moves first by randomly assigning a type $\theta_n \in \Theta_n$ according to an independent distribution f_{Θ_n} for each of the N players, assumed to be common knowledge. Each player observes their own type, but not those of the other players. Having observed their types, play continues in a sequence of stages $1, \ldots, T$ where at each time t every player n selects an action a_n^t. We define a history at time t to be the sequence of actions taken by all players up to that time, $h^t = (\mathbf{a}^1, \ldots, \mathbf{a}^{t-1})$. h^t is common knowledge for all t. After each history h^t each player may select an action from the set $\mathcal{A}_n(h^t)$ (possibly $\emptyset$) according to their strategy $s_n(h^t, \theta_n) \in \mathcal{P}(\mathcal{A}_n(h^t))$, where $\mathcal{P}(S)$ denotes the set of all distribution functions over a given set S. Thus strategies may in general be pure or mixed. Finally, we note that after history h^t all players share a common belief about player n's type, which we may denote by $\mu_n(\theta_n|h^t)$.

The solution concept that we consider for this class of games is a *perfect Bayesian-Nash equilibrium (PBNE)*. A pair of strategies and beliefs $(\mathbf{s}, \mu)$ is a PBNE if the following four conditions are met:

1. *Correct initial beliefs*: All players' beliefs are consistent with f_{θ_n}, i.e. $\mu_n(\theta_n|\emptyset) = f_{\theta_n}(\theta_n)\ \forall\ n, \theta_n$.

2. *Bayesian updating*: Suppose there exists some $\hat{\theta}_n \in \Theta_n$ such that $\mu_n(\hat{\theta}_n|h^t) > 0$ and $s_n(h^t, \hat{\theta}_n)(a_n^t) > 0$ for some action profile $\mathbf{a}^t$. Then,

$$\mu_n(\theta_n|h^t, \mathbf{a}^t) = \frac{\mu_n(\theta_n|h^t)s_n(h^t, \theta_n)(a_n^t)}{\int_{\Theta_n} \mu_n(\theta_n'|h^t)s_n(h^t, \theta_n')(a_n^t)d\theta_n'}\ \forall\ \theta_n \in \Theta_n.$$

3. *Action-determined beliefs*: If $a_n^t = \hat{a}_n^t$ for two action profiles $\mathbf{a}^t$ and $\hat{\mathbf{a}}^t$, then $\mu_n(\theta_n|h^t, \mathbf{a}^t) = \mu_n(\theta_n|h^t, \hat{\mathbf{a}}^t)$.

4. *Sequential rationality*: Player n's expected payoff starting after any history h^t, $t = 1, \ldots, T$ is maximized by playing strategy s_n.

3 2-Stage Auction Game

In our scenario there are two groups of players, the seller and the N brokers. The game begins with nature moving first by randomly assigning a type q to the seller according to the density function f_q and a type θ_n to each of the brokers independently according to a common density f_θ. Knowing their type q, the seller then makes a choice of which auction mechanism to employ and does so by selecting an action $a_s = (\mathcal{M}, \mathcal{T})$. We denote the seller's strategy by $s(q) \in \mathcal{P}(\{(I, q), (S, \emptyset)\})$. This action indicates two things: the selected auction mechanism, $\mathcal{M} \in \{S, I\}$, and their type, $\mathcal{T} \in \{q, \emptyset\}$. Note that $\mathcal{T} = \emptyset$ is associated with the standard auction and indicates that no type information is revealed. In addition, when intermediation is selected, $\mathcal{T} = q$ is revealed but only to the intermediary, acting on behalf of all the players, and not to the actual brokers themselves. Once the seller chooses their action, the mechanism proceeds down one of two paths. If $\mathcal{M} = S$ then the standard auction is employed and brokers compute their bids and a first-price sealed-bid auction is conducted, the highest bidding broker being awarded the portfolio. In this case no intermediary is used. In contrast, if $\mathcal{M} = I$ then intermediation is employed and the intermediary receives both the seller's private signal q, not revealed to the brokers, as well as bidding functions $\beta_{\mathrm{I}}^{n}(\theta_n, \cdot, N)$ from each of the brokers as explained in Padilla and Van Roy. The intermediary then computes the brokers' bids and again a first-price auction is run, the highest bidding broker being awarded the portfolio. Note that this is a *signaling game* in the sense that the seller functions as a "sender" whose message (their action a_s) is observed by the N brokers who collectively function as the "receiver", each of whom takes a simultaneous action (forming their bid) which determines all players' payoffs.

We may slightly reinterpret our game in a more convenient manner with one observation. With regards to broker behavior and thus the outcome of the auction, having an intermediary act on behalf of the brokers is equivalent to having no intermediary and communicating q to them directly. Although this is not how an intermediated auction would be conducted as it would defeat the goal of information confidentiality, for analysis purposes we may assume that it is done in this manner.

With this observation we may reinterpret and define our game of interest as follows:

Definition *(2-Stage Auction Game)* The *2-stage auction game* is a signaling game composed of a single seller and N brokers with play conducted as follows:

1. A single seller and N brokers come to the market and nature randomly draws types q for the seller and θ_n for each of the brokers. q is drawn according to a density f_q and each θ_n is drawn according to a density f_θ, $\Theta_n = \Theta\ \forall n$.

2. The seller chooses an action $a_s \in \{(I, q), (S, \emptyset)\}$ indicating the selected mechanism and their type. Here $\emptyset$ indicates that no type information is revealed by the seller.

3. Having observed the seller's action a_s, brokers compute and submit their bids.

4. A first-price sealed-bid auction is run, the highest bidding broker receiving the seller's portfolio in its entirety.

Note that the 2-stage auction game is an instance of a Bayesian multistage game with observable actions and independent types.

4 Intermediation as an Equilibrium

With this framework, we may now show that given a choice of standard and intermediated auction mechanisms that it is a PBNE for all seller types to select intermediation.

Claim 4.1. (Intermediation is a PBNE for the 2-Stage Auction Game) *Let* $f_q = unif([0,1])$ *and* $f_\theta = unif([0,1])$. *The following separating strategy is then a PBNE for the 2-stage auction game: All seller types select* $a_s = (I, q)$ *and have a belief about broker types given by* $\mu_b(\theta_n|\emptyset) = f_\theta(\theta_n)\ \forall n$. *Brokers' common prior belief of the seller's type is given by* $\mu_s(q|\emptyset) = f_q(q)$. *On the equilibrium path when* $a_s = (I, q)$, *brokers update their prior to a singleton at* q *and each* n*th broker with type* θ_n *submits a bid given by* $\beta_I(\theta_n, q, N)$. *Off the equilibrium path when* $a_s = (S, \emptyset)$, *brokers update their priors to a singleton at* $q = 1$ *and each* n*th broker with type* θ_n *submits a bid given by* $\beta_I(\theta_n, 1, N)$.

Proof. We need to show that these strategies and beliefs follow Properties 1-4 in the definition of a PBNE. We see that players' initial beliefs are consistent with f_q and f_θ, and thus Property 1 is immediately met. Furthermore, we see that players only use the actions of certain players to update their beliefs about those certain players; the actions of other players, given the same history, does not effect their beliefs. This applies specifically to broker beliefs about the seller's type and satisfies Property 3. Regarding Property 4, we see that given broker beliefs sellers of all types $q \in [0, 1)$ are better off selecting $a_s = (I, q)$, as selecting $a_s = (S, \emptyset)$ will prompt brokers to assume that the seller's type is the worst type possible ($q = 1$) and subsequently bid the worst values possible. Sellers of type $q = 1$ are indifferent and we make the assumption that they choose intermediation as well. Given either seller action, posterior broker beliefs about the seller's type will be a singleton at either the type provided by the seller (if $a_s = (I, q)$), or at $q = 1$ (if $a_s = (S, \emptyset)$). Either case is equivalent to the scenario of a broker computing an optimal bid in an intermediated auction where the seller's type is believed to take a certain value and the appropriate bidding function was derived in Padilla and Van Roy. Thus, players are sequentially rational and Property 4 is satisfied. Finally, we observe that the only Bayesian updating that occurs is when brokers update their common prior to a common posterior given the observed seller action. If the seller's action is $a_s = (S, \emptyset)$ then play is off the equilibrium path and Bayes' rule cannot be applied and instead brokers' beliefs that $q = 1$ is made by assumption. However, if the seller's action is $a_s = (I, q)$, then the seller's true type has been revealed and brokers update their belief appropriately to a singleton at that value, satisfying Property 2. Thus, all required properties are satisfied and the stated strategies and beliefs form a PBNE. □

As implied by the indifference of seller type $q = 1$ in the proof of Claim 4.1, this PBNE is not unique but rather is one of a family of PBNE for the 2-stage auction game. However, as shown in our second claim, all PBNE are identical up to the strategy of the seller with type $q = 1$, who may arbitrarily mix between the standard and intermediated mechanisms.

Claim 4.2. (Characterization of PBNE for the 2-stage Auction Game) *Let $f_q = unif([0,1])$ and $f_\theta = unif([0,1])$. The set of PBNE for the 2-stage auction game is the set of strategy profiles adhering to Claim 4.1 with the exception that the strategy of seller type $q = 1$ may mix between actions (I, q) and $(S, \emptyset)$ according to any arbitrary distribution.*

Proof. First, we noted in the proof of Claim 4.1 that sellers of type $q = 1$ are indifferent between both (I, q) and $(S, \emptyset)$ and hence any strategy for sellers of that type that mix between them according to any arbitrary distribution is also a PBNE. We now consider all other possible PBNEs that are outside this family and realize that any such candidate PBNE must fall into one of two cases. We will take each case in turn and show that any PBNE meeting the conditions of that case leads to a contradiction.

Case 1: A PBNE $(\mathbf{s}, \mu)$ such that $\exists\ q' \neq 1$ s.t. $s(q')((S, \emptyset)) > 0$ and $s(q)((I, q)) = 1\ \forall q \neq q'$.
In Case 1 the corresponding broker beliefs are such that $\mu_s(q = q'|(S, \emptyset)) = 1$ and thus all seller types $\hat{q}$ s.t. $q' < \hat{q} \leq 1$ are better off by playing action $(S, \emptyset)$ as this will lead to brokers believing the seller's type to be $\hat{q} > q'$, leading to strictly larger broker bids and higher expected seller revenue since each nth broker's optimal bidding function $\beta_I(\theta_n, q, N)$

is strictly monotonically decreasing with q. Hence, the PBNE adhering to Case 1 could not have been a best response for those seller types, giving us a contradiction.

Case 2: A PBNE $(\mathbf{s}, \mu)$ *such that* $\exists$ *a non-empty and non-singleton set* $\mathcal{S}_q$ *s.t.* $s(q')((S, \emptyset)) > 0 \ \forall q' \in \mathcal{S}_q$*.*

Let $\underline{q} = \inf(\mathcal{S}_q)$. We first assume that $\underline{q} \in \mathcal{S}_q$. We would like to show that for any (common) broker belief $\mu_s(\cdot, (S, \emptyset))$ over $q \in \mathcal{S}_q$ in an equilibrium in Case 2 that seller type $\underline{q}$ is strictly better off by always selecting action (I, q). We first note that the interim game[1] in which $(I, \underline{q})$ was first played by the seller is equivalent to the seller playing $(S, \emptyset)$ when broker beliefs place all probability weight on $q = \underline{q}$. Hence, to compare seller type $\underline{q}$'s expected revenue under an equilibrium in Case 2 with the expected revenue they would instead receive if they were to deviate to only playing $(I, \underline{q})$, we may equivalently compare an interim game in which the seller plays $(S, \emptyset)$ under two different corresponding broker beliefs about q, $\mu_s(\cdot, (S, \emptyset))$ and $\mathbb{1}_{[q=\underline{q}]}(\cdot)$ respectively. Given Assumptions 1.1 and 1.2, it may be shown that the beliefs of each broker n regarding the certainty equivalent value $v_{ce}(\theta_j)$ of each broker $j \neq n$ under belief $\mathbb{1}_{[q=\underline{q}]}(\cdot)$ first-order stochastically dominates those under belief $\mu_s(\cdot, (S, \emptyset))$. Given these facts, it may be shown that this interim game is a monotone supermodular game as defined in Van Zandt and Vives (2007). They show that if a monotone supermodular game meets certain technical conditions, which our game may be shown to satisfy, then if each player's interim beliefs shift with respect to first-order stochastic dominance, then the greatest and least Bayesian-Nash equilibria strictly increase (Corollary 17). As there is only a single Bayesian-Nash equilibrium in our game, as established in Padilla and Van Roy, bidding functions for each broker under belief $\mathbb{1}_{[q=\underline{q}]}(\cdot)$ are *strictly* greater than those under belief $\mu_s(\cdot, (S, \emptyset))$. Hence, the expected revenue for seller type $\underline{q}$ is *strictly* higher under action $(I, \underline{q})$ and the assumed equilibrium in which seller type $\underline{q}$ plays $(S, \emptyset)$ with non-zero probability could not have been a best response and we have a contradiction. Now, we consider $\underline{q} \notin \mathcal{S}_q$. As $\underline{q}$ is a limit point of the set $\mathcal{S}_q$, there exists a $\tilde{\underline{q}}$ within any arbitrarily small ϵ-neighborhood of $\underline{q}$ s.t. $\tilde{\underline{q}} \in \mathcal{S}_q$. We may thus proceed in the same fashion, considering a point $\tilde{\underline{q}}$ arbitrarily close to $\underline{q}$ and similarly conclude that the expected revenue for seller

[1] The game played by the brokers in steps 3 and 4 of the 2-stage auction game after the seller takes their action.

type $\underline{\tilde{q}}$ is higher under action $(I, \underline{\tilde{q}})$ and hence the assumed equilibrium in which seller type $\underline{\tilde{q}}$ plays $(S, \emptyset)$ with non-zero probability could not have been a best response, again giving us a contradiction. □

Because in any PBNE all brokers know the specific type $q = 1$ of the seller that may potentially mix between the two actions, they thus trivially know the true value of that seller's type if the action $(S, \emptyset)$ is played, making the choice irrelevant. Hence, in effect all PBNE's are equivalent to the fully separating equilibrium stated in Claim 4.1 in which all seller types select action (I, q), selecting intermediation and providing their type to the intermediary. These results demonstrate that under the assumptions made intermediation is a stable operating point for market participants and that the computational studies discussed in Padilla and Van Roy can be predicted to hold in practice as sellers of all types q would indeed select the intermediated mechanism were that option available in addition to the current standard mechanism.

References

D. Fudenberg and J. Tirole. Perfect bayesian equilibrium and sequential equilibrium. *Journal of Economic Theory*, 53, 1991.

J. Harsanyi. Games with incomplete information played by bayesian players, parts i, ii, and iii. *Management Science*, 14, 1967.

M. Osborne and A. Rubinstein. *A Course in Game Theory*. The MIT Press, 1994.

M. Padilla and B. Van Roy. Intermediated blind portfolio auctions. *Management Science (to appear)*.

T. Van Zandt and X. Vives. Monotone equilibria in bayesian games of strategic complementarities. *Journal of Economic Theory*, 134(1):339–360, 2007.